

# The spinor equation for the electromagnetic field

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February 1, 2008

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## ABSTRACT

We develop a spinor equation of the electromagnetic field, which is equivalent to the Maxwell equation and has a similar form as the Dirac equation. The spinor is the very conjugate momentum of the vector potential in the Lagrangian mechanics. In this framework the electromagnetic field described by the spinor exhibits the  $SU(2)$  internal symmetry. The quantization and the problem of longitudinal and scalar photons are discussed.

PACS number(s): 41.20.Bt, 03.50.De

## Introduction

Conventionally, to describe the electromagnetic field one has introduced the electromagnetic tensor as well as the vector potential. However, the tensor has some redundant components and the form of the field equation is not as simple and compact as the Dirac equation, which can not only give us inconvenience in applications but also hinder us from revealing the internal symmetry of the system.

In this paper we introduce a spinor in replace of the previous tensor to describe the electromagnetic field, by the help of which the Maxwell equation is equivalent to the spinor equation which has the similar form as the Dirac equation. The spinor happen to be the conjugate momentum of the vector potential. We also exploit the internal symmetry of the field.

In section I we give the classical form of the spinor equation of the field. In section II we study the Fourier property of the free field. In section III we showed the Lorentz invariance of the equation. In section IV we give the Lagrangian of the field and study the symmetry. In Section V we study the quantization of the field in the Lagrangian mechanics.

# 1 The new wave equation for the electromagnetic field

As is know that the Maxwell equation can be expressed in the complex form as

$$\begin{aligned}\nabla \times \vec{\phi} - \partial_4 \vec{\phi} &= \sqrt{\frac{\mu_0}{2}} \vec{j}_e \\ \nabla \cdot \vec{\phi} &= \sqrt{\frac{\mu_0}{2}} j_{e4}\end{aligned}\quad (1.1)$$

where

$$\vec{\phi} = \frac{1}{\sqrt{2\mu_0}} (\vec{B} + i \vec{E})$$

is a three-component vector in the coordinate space, and  $j_e$  is the four-component electric current vector.

If we consider  $\phi$  a three-component column and add a fourth component to it, we get a four-component column

$$\phi \equiv \begin{pmatrix} \vec{\phi} \\ \xi \end{pmatrix} \quad (1.2)$$

Using Eq.(1.1) we can construct a wave equation of  $\phi$  defined in Eq.(1.2) as

$$\Lambda_\mu \partial_\mu \phi = \sqrt{\frac{\mu_0}{2}} j_e \quad (1.3)$$

where

$$\begin{aligned}\Lambda_1 &= \begin{pmatrix} 0 & 0 & 0 & -1 \\ 0 & 0 & -1 & 0 \\ 0 & 1 & 0 & 0 \\ 1 & 0 & 0 & 0 \end{pmatrix}, & \Lambda_2 &= \begin{pmatrix} 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & -1 \\ -1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \end{pmatrix}, \\ \Lambda_3 &= \begin{pmatrix} 0 & -1 & 0 & 0 \\ 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & -1 \\ 0 & 0 & 1 & 0 \end{pmatrix}, & \Lambda_4 &= -I\end{aligned}$$

are unitary matrices.

Considering that  $j_e$  is a conserved current of the Dirac field, we get that  $\xi$  satisfies the Klein-Gordon equation

$$\partial_\mu \partial_\mu \xi = 0. \quad (1.4)$$

As  $j_e$  is a Lorentz vector  $\xi$  can be proved to be a Lorentz scalar. Therefore  $\xi$  can have a zero solution

$$\xi(\vec{x}, t) = 0 \text{ (for any } \vec{x} \text{ and } t). \quad (1.5)$$

Apperantly, Eq.(1.3) is completely equivalent to the Maxwell equation (1.1) under the condition Eq.(1.5). We adopt this condition in this and next section.

Eq.(1.5) will be called the transversality condition, which is the requirement of Maxwell equation to Eq.(1.3).

One can easily check that  $\phi$  can be described in terms of the electromagnetic potential  $A$  as

$$\phi = -\frac{1}{\sqrt{2\mu_0}}\Lambda_\mu^{-1}\partial_\mu A \quad (1.6)$$

From Eq.(1.6) we know that Eq.(1.5) produces the Lorentz condition

$$\partial_\mu A_\mu = 0. \quad (1.7)$$

The usual four-component electromagnetic energy flux density can be expressed as

$$j_\mu = -i\phi^+\Lambda_\mu\phi$$

Therefore the Maxwell equation is now described in Eq.(1.3) plus the condition Eq.(1.5), which is equivalent to Lorentz condition.

## 2 The Fourier decomposition of the free field

In this section we discuss the properties of the free field on the condition Eq.(1.5). we will give the plane wave solution and show the Lorentz invariance of the field.

For free field Eq.(1.3) reduces to

$$\partial_\mu \Lambda_\mu \phi = 0 \quad (2.1)$$

which has the similar form as Dirac Equation.

The plane wave solution to Eq.(2.1) can be expressed as

$$\phi_{\vec{k},\alpha}^{\rightarrow}(\vec{x},t) = C_{\vec{k},\alpha}^{\rightarrow} u_{\vec{k},\alpha}^{\rightarrow} e^{i(\vec{k}\cdot\vec{x}-\omega t)} \quad (2.3)$$

where  $C_{\vec{k},\alpha}^{\rightarrow}$  is a constant,  $u_{\vec{k},\alpha}^{\rightarrow}$  is a unit four-component column, satisfying

$$u_{\vec{k},\alpha}^{\rightarrow\dagger} u_{\vec{k},\alpha'}^{\rightarrow} = \delta_{\alpha,\alpha'}$$

and  $\alpha$  is the energy sign parameter, satisfying

$$\omega = \alpha |\vec{k}| \quad (2.4)$$

where  $\alpha = \pm 1$ . Any electromagnetic field can be expanded on the basis of the plane wave solution as in Eq.(2.3).

By virtue of Eq.(1.5) one obtains

$$\begin{aligned} u_{\vec{k},\alpha,4}^{\rightarrow} &= 0 \\ \vec{k} \cdot \vec{u}_{\vec{k},\alpha}^{\rightarrow} &= 0 \end{aligned} \quad (2.5)$$

which states that the fourth component of  $u_{\vec{k},\alpha}$  is zero and the first three components are transverse. Therefore the condition (1.5) as well as the Lorentz condition(1.7) is the transversality condition. We should note that without the transversality condition Eq.(2.1) has other solutions that do not satisfy Eq.(2.5) and can be viewed as the longitudinal and scalar field.

### 3 The spinor equation and the helicity of the combined field

In this section we drop the constraint condition Eq.(1.5) attached to Eq.(1.3) and see what we get. We can suppose the Eq.(1.3) describe the combined field: electromagnetic field plus a scalar field. After careful check we will find the  $\phi$  defined in Eq.(1.2) is a spinor and  $\xi$  is a scalar.

Considering that  $j_e$  in Eq.(1.3) is a Lorentz vector, we can prove that when the system undergoes the normal Lorentz transformations,  $\phi$  transforms as

$$\phi' = T\phi$$

where

$$T = -\Lambda_\mu^{-1} a_{\mu 4} a \quad (3.1)$$

and the form of Eq.(1.3) keeps invariant in the new frame. Therefore  $\phi$  is a spinor and Eq.(2.1) is the spinor equation for the free combined field, which is comparable to Dirac equation. From Eq.(3.1) one can get that  $\xi$  is a scalar, which provides the justification for the condition (1.5).

From Eq.(3.1) the spin matrices are found to be

$$s_1 = \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & -i & 0 \\ 0 & i & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}, s_2 = \begin{pmatrix} 0 & 0 & i & 0 \\ 0 & 0 & 0 & 0 \\ -i & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}, s_3 = \begin{pmatrix} 0 & -i & 0 & 0 \\ i & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix} \quad (3.2)$$

,which indicates that the field has spin 1.

The field helicity can be defined as

$$h \equiv \vec{s} \cdot \vec{p} / |\vec{p}|$$

which is conserved in time. We find that the plane wave as in Eq.(2.3) with the condition (2.5) has the  $-\alpha$  helicity

$$h\phi_{\vec{k},\alpha} = -\alpha\phi_{\vec{k},\alpha} \quad (3.3)$$

We should note that, without the constraint condition Eq.(1.5), Eq.(2.1) has a zero helicity solution  $\phi_{\vec{k},\alpha}^L$  satisfying

$$h\phi_{\vec{k},\alpha}^L = 0.$$

We must note that  $\phi_{k,\alpha}^l$  is beyond the extent of the actual electromagnetic field. In fact one can check that  $\phi_{k,\alpha}^l$  represents the longitudinal or scalar field while  $\phi_{k,\alpha}^\rightarrow$  represents the transverse field. It is the Maxwell equation that rules out the possibility of longitudinal and scalar fields.

In this section we show that Eq.(1.3) is invariant under Lorentz transformation and  $\phi$  is a spinor.

## 4 The Lagrangian and the internal symmetry of the field

In fact Eq.(1.5) is the classic express of the transversality condition. In this section in order to prepare for the quantization we drop the constraint condition Eq.(1.5). We study some properties of the combined field without the condition(1.5) in Lagrangian mechanics in this section.

We seek a generalized potential  $A$  which is free of Lorentz condition and satisfies Eq.(1.6). The fourth component in Eq.(1.6) reads

$$\xi = \frac{1}{\sqrt{2\mu_0}} \partial_\mu A_\mu. \quad (4.1)$$

From Eq.(1.4) we know that the field  $\xi$  is a massless Klein-Gordon field. The Lorentz condition as in Eq.(1.7) no longer holds.

The Lagrangian density of the combined field can be defined by

$$L = -\phi^2 \quad (4.2)$$

We suppose  $A$  is the canonical coordinate of the field and we get the conjugate momenta

$$\pi_\mu = i\sqrt{\frac{2}{\mu_0}} \phi_\mu.$$

Therefore  $\phi$  is the conjugate momenta of the canonical coordinate  $A$  except for a factor. The Hamiltonian density is found to be

$$H = \frac{1}{\sqrt{2\mu_0}} \phi \Lambda_\mu \partial_\mu A. \quad (4.3)$$

By studying Eq.(4.2) one can find the system in free state has an internal symmetry. There exists an isospin operator

$$\vec{s}^{iso} = \frac{i}{2} \vec{\Delta}$$

where

$$\Delta_1 = \begin{pmatrix} 0 & 0 & 0 & 1 \\ 0 & 0 & -1 & 0 \\ 0 & 1 & 0 & 0 \\ -1 & 0 & 0 & 0 \end{pmatrix}, \quad \Delta_2 = \begin{pmatrix} 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ -1 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 \end{pmatrix},$$

$$\Delta_3 = \begin{pmatrix} 0 & -1 & 0 & 0 \\ 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & -1 & 0 \end{pmatrix}$$

are unitary matrices. The corrspring currents read

$$j_\mu^i = \pi \Lambda_\mu^{-1} \Delta_i A$$

which is conserved for the free field. However the symmetry will break on the interaction with other fields such as the Dirac field.

## 5 The quantization of the field

In this section we study the quatization of the combined field in the framework of last section. In order to deal with the quantization of the field we introduce a set of normal orthogonal basis in the four-dimension space

$$\epsilon_{k1}^{\rightarrow} = \begin{pmatrix} \vec{e}_1 \\ 0 \end{pmatrix}, \quad \epsilon_{k2}^{\rightarrow} = \begin{pmatrix} \vec{e}_2 \\ 0 \end{pmatrix}, \quad \epsilon_{k3}^{\rightarrow} = \begin{pmatrix} \vec{k}^0 \\ 0 \end{pmatrix}, \quad \epsilon_{k4}^{\rightarrow} = \begin{pmatrix} 0 \\ 1 \end{pmatrix}$$

where  $\vec{e}_1, \vec{e}_2, \vec{k}^0$  are normal orthogonal basis in the three-dimension space. Considering that  $\vec{A}, iA_4$  are Hermitian operator, we need to define an accompanying set of basis as

$$\bar{\epsilon}_{ki}^{\rightarrow} = \epsilon_{ki}^{\rightarrow}$$

$$\bar{\epsilon}_{k4}^{\rightarrow} = -\epsilon_{k4}^{\rightarrow}$$

for i=1,2,3. The canonical coordinate operator A can be expressed as

$$A(\vec{x}, t) = \sqrt{\frac{\mu_0}{2V}} \sum_{\vec{k}} \sum_{\lambda=1}^4 \frac{1}{\sqrt{|\vec{k}|}} [\epsilon_{k\lambda}^{\rightarrow} a(t)_{k\lambda}^{\rightarrow} e^{i\vec{k} \cdot \vec{x}} + \bar{\epsilon}_{k\lambda}^{\rightarrow} a(t)_{k\lambda}^{\rightarrow \dagger} e^{-i\vec{k} \cdot \vec{x}}] \quad (5.1)$$

where  $a(t)_{k\lambda}^{\rightarrow}$  is the annihilation operator, satisfying

$$[a(t)_{k\lambda}^{\rightarrow}, a(t)_{k'\lambda'}^{\rightarrow \dagger}] = \delta_{k',k}^{\rightarrow} \delta_{\lambda',\lambda}. \quad (5.2)$$

From Eq.(1.6) and Eq.(5.1) we get the momenta operator

$$\phi(\vec{x}, t) = \frac{i}{\sqrt{2V}} \sum_{\vec{k}\lambda} \sqrt{|\vec{k}|} [v_{k\lambda}^{\rightarrow} a(t)_{k\lambda}^{\rightarrow} e^{i\vec{k} \cdot \vec{x}} - \bar{v}_{k\lambda}^{\rightarrow} a(t)_{k\lambda}^{\rightarrow \dagger} e^{-i\vec{k} \cdot \vec{x}}] \quad (5.3)$$

where

$$\begin{aligned} v_{\vec{k}\lambda} &= -\frac{1}{\sqrt{2}|\vec{k}|} k_\mu \Lambda_\mu^{-1} \epsilon_{\vec{k}\lambda} \\ \bar{v}_{\vec{k}\lambda} &= -\frac{1}{\sqrt{2}|\vec{k}|} k_\mu \Lambda_\mu^{-1} \bar{\epsilon}_{\vec{k}\lambda} \end{aligned}$$

From Eq.(5.1), Eq.(5.2) and Eq.(5.3) we can get that the canonical coordinate  $A$  and it's momenta  $\pi$  satisfy the standard commutation relation

$$[A_\mu(\vec{x}, t), \pi_\nu(\vec{x}', t)] = i\delta_{\mu\nu}\delta^3(\vec{x} - \vec{x}')$$

From Eq.(4.3), (5.1) and (5.3) we get the Hamiltonian of the combined field

$$H = \sum_{\vec{k}} |\vec{k}| \left[ \sum_{\lambda=1}^2 (N_{\vec{k}\lambda} + \frac{1}{2}) + N_{\vec{k}3} - N_{\vec{k}4} \right] \quad (5.4)$$

where  $N_{\vec{k}\lambda}$  is number operator. The momentum operator of the field can be calculated as well

$$\vec{P} = \sum_{\vec{k}} \vec{k} \left[ \sum_{\lambda=1}^2 (N_{\vec{k}\lambda} + \frac{1}{2}) + N_{\vec{k}3} - N_{\vec{k}4} \right] \quad (5.5)$$

Eq.(5.4) and Eq.(5.5) is consistent with the conventional result. The problem of negative energy as in Eq.(5.4) will be solved as follow.

With the help of Eq.(4.1) and (5.1) we get the fourth component of  $\phi$  in Eq.(5.3)

$$\xi = \frac{i}{\sqrt{2V}} \sum_{\vec{k}} \sqrt{|\vec{k}|} [b_{\vec{k}} e^{i\vec{k} \cdot \vec{x}} - h.c.]$$

,where we define a new annihilation operator as

$$b_{\vec{k}} = \frac{1}{\sqrt{2}} (a_{\vec{k}3} + i a_{\vec{k}4})$$

Now we turn back to the transversality condition (1.5). The  $\phi$  in Eq.(1.5) should be interpreted as the average of the operator in a state. The condition (1.5) turns to be

$$\langle a | \xi | a \rangle = 0, \quad (5.6)$$

where  $|a\rangle$  is an arbitrary state. Eq.(5.6) implies that the operator  $b$  should annihilate any state i.e.

$$b_{\vec{k}} |a\rangle = 0 \quad (5.7)$$

With the constraint condition Eq.(5.7),  $N_{\vec{k}3} - N_{\vec{k}4}$  in Eq.(5.4) and (5.5) have no actual contribution.

In summary, the electromagnetic field described by Eq.(1.3) is well quantized with the condition Eq.(5.6) or Eq.(5.7).

## CONCLUSION

We rewrite the Maxwell equation into an equivalent spinor equation which is Lorentz invariant and has a Dirac-like form. The condition that the fourth component of the spinor is zero or the transversality condition is necessary and feasible, which is equivalent to the Lorentz condition. On quantization in the Lagrangian mechanics, the fourth component is a nonvanishing operator and the transversality condition reduces to the vanishing of the average of the operator in all states. The appearance of the longitudinal and scalar fields are the requirement of the symmetry of the field and the null observable contribution of them is the result of the transversality condition. The spinor also plays a definite role in Lagrangian mechanics: it is the conjugate momentum of the vector potential, so the spinor and the vector are two corresponding and necessary parts of the field. From the Lagrangian of the field we find there exists the  $SU(2)$  symmetry of the field and we calculate the conserved currents. The field is well depicted in a symmetric and simple form by the employment of the spinor in place of the tensor, which can not only simplify the calculation but also reveal the internal symmetry, and which will find more application.

## REFERENCES

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